EHRHART POLYNOMIALS, SIMPLICIAL POLYTOPES, MAGIC SQUARES AND A CONJECTURE OF STANLEY

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Dedicated to Richard Stanley on the occasion of his sixtieth birthday

Abstract. It is proved that a certain symmetric sequence $(h_0, h_1, \ldots, h_d)$ of nonnegative integers arising in the enumeration of magic squares of given size n by row sums or, equivalently, in the generating function of the Ehrhart polynomial of the polytope of doubly stochastic $n \times n$ matrices, is equal to the h-vector of a simplicial polytope and hence that it satisfies the conditions of the g-theorem. The unimodality of $(h_0, h_1, \ldots, h_d)$, which follows, was conjectured by Stanley (1983). Several generalizations are given.

1. Introduction

A *magic square* is a square matrix with nonnegative integer entries having all line sums equal to each other, where a line is a row or a column. Let $H_n(r)$ be the number of $n \times n$ magic squares with line sums equal to r. The problem to determine $H_n(r)$ appeared early in the twentieth century [10]. Since then it has attracted considerable attention within areas such as combinatorics, combinatorial and computational commutative algebra, discrete and computational geometry, probability and statistics [1, 2, 4, 5, 8, 16, 17, 19, 23, 24, 26]. It was conjectured by Anand, Dumir and Gupta [1] and proved by Ehrhart [5] and Stanley [17] (see also [23, Section I.5] and [24, Section 4.6]) that for any fixed positive integer n, the quantity $H_n(r)$ is a polynomial in r of degree $(n-1)^2$. More precisely, the following theorem holds.

Theorem 1.1. (Stanley [17, 19]) *For any positive integer n we have*

$$\sum_{r \geq 0} H_n(r)\, t^r = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{(n-1)^2+1}}, \tag{1}$$

where $d = n^2 - 3n + 2$ and the h_i are nonnegative integers satisfying $h_0 = 1$ and $h_i = h_{d-i}$ for all i.

It is the first conjecture stated in [23] (see Section I.1 there) that the integers h_i appearing in (1) satisfy further the inequalities

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}. \tag{2}$$

In this paper we prove this conjecture by showing that $(h_0, h_1, \ldots, h_d)$ is equal to the h-vector of a d-dimensional simplicial polytope. Such vectors are known to be symmetric and unimodal and are characterized by McMullen's g-theorem [11]; see [3, 21] and Section 2.

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A few comments on Stanley's conjecture and the proof given in this paper are in order. It is known that the sequence $(h_0, h_1, \ldots, h_d)$ of Theorem 1.1 is a *Gorenstein sequence*, meaning it is the h-vector of a standard, graded, Gorenstein commutative ring; see for instance [23, Section I.13]. It is an important open problem to characterize Gorenstein sequences; see [23, Section II.6]. In this direction it was originally conjectured by Stanley [18] that a sequence $(h_0, h_1, \ldots, h_d)$ is Gorenstein if and only if it satisfies the conditions of the g-theorem but a counterexample was later given in [20]. Our result (Corollary 3.6) and its generalization to the enumeration of magic labelings of regular bipartite graphs (Corollary 4.5) give an instance in which Stanley's original conjecture turns out to be true. Another such instance, in which the entries of $(h_0, h_1, \ldots, h_d)$ count linear extensions of a naturally labeled poset by the number of descents, was given recently by Reiner and Welker [15]. The polynomial $H_n(r)$ is the Ehrhart polynomial (see Section 2) of the Birkhoff polytope of doubly stochastic $n \times n$ matrices. We will state our main result in the context of Ehrhart polynomials of integer polytopes (Theorem 3.5) as well as that of enumerating solutions to systems of linear homogeneous Diophantine equations (Corollary 4.1) and will show that both situations of Theorem 1.1 and [15] appear as special cases.

This paper was largely motivated by the work [15] of Reiner and Welker. I am grateful to Volkmar Welker for encouraging discussions and to Jesús DeLoera, Victor Reiner, Francisco Santos and Richard Stanley for helpful suggestions.

2. Background

In this section we review some basic definitions and background on convex polytopes and their face numbers, triangulations and Ehrhart polynomials. We refer the reader to the texts by Stanley [23, 24], Sturmfels [27] and Ziegler [28] for more information on these topics. We denote by $\mathbb{N}$ the set of nonnegative integers.

Face enumeration. Given a finite (abstract or geometric) simplicial complex Δ of dimension $d-1$, let f_i denote the number of i-dimensional faces of Δ, so that $(f_0, f_1, \ldots, f_{d-1})$ is the *f-vector* of Δ. The polynomial

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}, \tag{3}$$

where $f_{-1} = 1$ unless Δ is empty, is the *h-polynomial* of Δ, denoted $h(\Delta, t)$. The *h-vector* of Δ is the sequence $(h_0, h_1, \ldots, h_d)$ defined by (3).

A *polytopal complex* $\mathcal{F}$ [28, Section 8.1] is a finite, nonempty collection of convex polytopes such that (i) any face of a polytope in $\mathcal{F}$ is also in $\mathcal{F}$ and (ii) the intersection of any two polytopes in $\mathcal{F}$ is either empty or a face of both. The elements of $\mathcal{F}$ are its *faces* and those of dimension 0 are its *vertices*. The *dimension* of $\mathcal{F}$ is the maximum dimension of a face. The complex $\mathcal{F}$ is *pure* if all maximal faces of $\mathcal{F}$ have the same dimension. The collection $\mathcal{F}(P)$ of all faces of a polytope P and the collection $\mathcal{F}(\partial P)$ of its proper faces are pure polytopal complexes called the *face complex* and *boundary complex* of P, respectively. Thus P is *simplicial* if $\mathcal{F}(\partial P)$ is a simplicial complex. The h-vectors of boundary complexes of simplicial polytopes are characterized by McMullen's *g-theorem* [11] [23, Section III.1] [28, Section 8.6] as follows. A sequence $(g_0, g_1, \ldots, g_\ell)$ of nonnegative integers is said to be an *M-vector* if

(i) $g_0 = 1$ and
(ii) $0 \le g_{i+1} \le g_i^{(i)}$ for $1 \le i \le \ell - 1$,

where $0^{(i)} = 0$ and

$$n^{(i)} = \binom{k_i+1}{i+1} + \binom{k_{i-1}+1}{i} + \cdots + \binom{k_j+1}{j+1}$$

for the unique representation

$$n = \binom{k_i}{i} + \binom{k_{i-1}}{i-1} + \cdots + \binom{k_j}{j}$$

with $k_i > k_{i-1} > \cdots > k_j \ge j \ge 1$, if $n \ge 1$. A sequence $(h_0, h_1, \ldots, h_d)$ of nonnegative integers is the h-vector of the boundary complex of a d-dimensional simplicial polytope if and only if

(i) $h_i = h_{d-i}$ for all i and
(ii) $(h_0, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-vector.

In particular $(h_0, h_1, \ldots, h_d)$ is symmetric and unimodal and hence satisfies the inequalities (2), known as the *Generalized Lower Bound Theorem* for simplicial polytopes.

Triangulations and Ehrhart polynomials. A *triangulation* of a polytopal complex $\mathcal{F}$ is a geometric simplicial complex Δ with vertices those of $\mathcal{F}$ and underlying space equal to the union of the faces of $\mathcal{F}$, such that every maximal face of Δ is contained in a face of $\mathcal{F}$. A triangulation of the face complex $\mathcal{F}(P)$ of a polytope P is simply called a triangulation of P.

For any set σ consisting of vertices of the polytopal complex $\mathcal{F}$ we denote by $\mathcal{F} \backslash \sigma$ the subcomplex of faces of $\mathcal{F}$ which do not contain any of the vertices in σ and write $\mathcal{F} \backslash v$ for $\mathcal{F} \backslash \sigma$ if σ consists of a single vertex v. Given a linear ordering $\tau = (v_1, v_2, \ldots, v_p)$ of the set of vertices of $\mathcal{F}$ we define the *reverse lexicographic triangulation* or *pulling triangulation* $\Delta(\mathcal{F}) = \Delta_\tau(\mathcal{F})$ with respect to τ [22] [9] [27, p. 67] as $\Delta(\mathcal{F}) = \{v\}$ if $\mathcal{F}$ consists of a single vertex v and

$$\Delta(\mathcal{F}) = \Delta(\mathcal{F} \backslash v_p) \cup \bigcup_F \{\mathrm{conv}(\{v_p\} \cup G) : G \in \Delta(\mathcal{F}(F)) \cup \{\emptyset\}\}$$

otherwise, where the union runs through the facets F not containing v_p of the maximal faces of $\mathcal{F}$ which contain v_p and $\Delta(\mathcal{F} \backslash v_p)$ and $\Delta(\mathcal{F}(F))$ are defined with respect to the linear orderings of the vertices of $\mathcal{F} \backslash v_p$ and F, respectively, induced by τ. Equivalently, for $i_0 < i_1 < \cdots < i_t$ the set $\{v_{i_0}, v_{i_1}, \ldots, v_{i_t}\}$ is the vertex set of a maximal simplex of $\Delta_\tau(\mathcal{F})$ if there exists a maximal flag $F_0 \subset F_1 \subset \cdots \subset F_t$ of faces of $\mathcal{F}$ such that v_{i_j} is the last vertex of F_j with respect to τ for all j and v_{i_j} is not a vertex of F_{j-1} for $j \ge 1$. A different way to define $\Delta_\tau(\mathcal{F})$ is the following. For any vertex v of $\mathcal{F}$ let

$$\mathrm{pull}_v(\mathcal{F}) = (\mathcal{F} \backslash v) \cup \bigcup_F \{\mathrm{conv}(\{v\} \cup G) : G \in \mathcal{F}(F) \cup \{\emptyset\}\},$$

where the union runs through the facets F not containing v of the maximal faces of $\mathcal{F}$ which contain v. If $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i = \mathrm{pull}_{v_{p-i+1}}(\mathcal{F}_{i-1})$ for $1 \le i \le p$ then $\mathcal{F}_p$ is a triangulation of $\mathcal{F}$ which coincides with $\Delta_\tau(\mathcal{F})$. It follows from [12, Theorem 2.5.23] (see also [6, p. 80]) that if $\mathcal{F}$ is the boundary complex of a polytope P then $\mathrm{pull}_v(\mathcal{F})$ is the boundary complex of another polytope, obtained from P by moving

its vertex v beyond the hyperplanes supporting exactly those facets of P which contain v. This observation implies the following lemma.

Lemma 2.1. *The reverse lexicographic triangulation of the boundary complex of a polytope with respect to any ordering of its vertices is abstractly isomorphic to the boundary complex of a simplicial polytope of the same dimension.*

A convex polytope $P \subseteq \mathbb{R}^q$ is said to be a *rational* or an *integer* polytope if all its vertices have rational or integer coordinates, respectively. It is called a *0-1 polytope* if all its vertices are 0-1 vectors in $\mathbb{R}^q$. If P is rational then the function defined for nonnegative integers r by the formula

$$\mathrm{Ehrhart}(P, r) = \#\,(rP \cap \mathbb{Z}^q)$$

is a quasi-polynomial in r, called the *Ehrhart quasi-polynomial* of P [24, Section 4.6]. If P is an integer polytope then this quasi-polynomial is actually a polynomial in r. Let $A \subseteq \mathbb{R}^q$ be the affine span of the integer polytope P. A triangulation Δ of P is called *unimodular* if the vertex set of any maximal simplex of Δ is a basis of the affine integer lattice $A \cap \mathbb{Z}^q$. We denote by Δ_τ the reverse lexicographic triangulation of an arbitrary polytope P with respect to the ordering τ of its vertices. Following [22] we call such an ordering of the vertices of an integer polytope P *compressed* if Δ_τ is unimodular and call P itself *compressed* if so is any linear ordering of its vertices. The following lemma holds for any unimodular triangulation of P, although we will not need this fact here.

Lemma 2.2. ([22, Corollary 2.5]) *If P is an m-dimensional integer polytope in $\mathbb{R}^q$ and τ is a compressed ordering of its vertices then*

$$\sum_{r \geq 0} \mathrm{Ehrhart}(P, r)\, t^r = \frac{h(\Delta_\tau, t)}{(1-t)^{m+1}}.$$

If $P \subseteq \mathbb{R}^m$ is an m-dimensional polytope and V is any linear subspace of $\mathbb{R}^m$ then the *quotient polytope* $P/V \subseteq \mathbb{R}^m/\,V$ is the image of P under the canonical surjection $\mathbb{R}^m \to \mathbb{R}^m/\,V$. This is a convex polytope in $\mathbb{R}^m/\,V$ linearly isomorphic to the image $\pi(P)$ of P under any linear surjection $\pi : \mathbb{R}^m \to \mathbb{R}^{m-\dim V}$ with kernel V. Recall that the simplicial join $\Delta_1 * \Delta_2$ of two abstract simplicial complexes Δ_1 and Δ_2 on disjoint vertex sets has faces the sets of the form $\sigma_1 \cup \sigma_2$, where $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$ and that $h(\Delta_1 * \Delta_2, t) = h(\Delta_1, t)\, h(\Delta_2, t)$. The following proposition is essentially Proposition 3.12 in [15].

Proposition 2.3. *Let P be an m-dimensional polytope in $\mathbb{R}^m$ having a triangulation abstractly isomorphic to $\sigma * \Delta$, where σ is the vertex set of a simplex not contained in the boundary of P. Let V be the linear subspace of $\mathbb{R}^m$ parallel to the affine span of σ.*

The boundary complex of the quotient polytope $P/V \subseteq \mathbb{R}^m/\,V$ is abstractly isomorphic to $\mathcal{F}(P) \setminus \sigma$ and inherits a triangulation abstractly isomorphic to Δ.

Two compressed polytopes. (a) A real $n \times n$ matrix is said to be *doubly stochastic* if all its entries are nonnegative and all its rows and columns sum to 1. The set P of all real doubly stochastic $n \times n$ matrices is a convex polytope in $\mathbb{R}^{n \times n}$ of dimension $(n-1)^2$, called the *Birkhoff polytope* [28, Example 0.12]. It follows from the classical Birkhoff-von Neumann theorem that the vertices of P are the $n \times n$

permutation matrices, so that P is a 0-1 polytope. The Birkhoff polytope was shown to be compressed by Stanley [22, Example 2.4 (b)] (see also [27, Corollary 14.9]).

(b) Let Ω be a poset (short for partially ordered set) on the ground set $[m] := \{1, 2, \dots, m\}$. Recall that an (order) ideal of Ω is a subset $I \subseteq \Omega$ for which $i <_\Omega j$ and $j \in I$ imply that $i \in I$. Let Ω^0 be the poset obtained from Ω by adjoining a minimum element 0. The *order polytope* [25] of Ω, denoted $O(\Omega)$, is the intersection of the hyperplane $x_0 = 1$ in $\mathbb{R}^{m+1}$ with the cone defined by the inequalities $x_i \geq x_j$ for $i < j$ in Ω^0 and $x_i \geq 0$ for all i. Thus $O(\Omega)$ is an m-dimensional convex polytope. The vertices of $O(\Omega)$ are the characteristic vectors of the nonempty ideals of Ω^0 [25, Corollary 1.3] so, in particular, $O(\Omega)$ is a 0-1 polytope (see [25, Theorem 1.2] for a complete description of the facial structure of $O(\Omega)$). Order polytopes were shown to be compresed by Ohsugi and Hibi [13, Example 1.3 (b)].

3. Special simplices

Throughout this section P denotes an m-dimensional convex polytope in $\mathbb{R}^q$ with face complex $\mathcal{F}(P)$. Let Σ be a simplex spanned by n vertices of P. We call Σ a *special simplex* in P if each facet of P contains exactly $n-1$ of the vertices of Σ. Note that, in particular, Σ is not contained in the boundary of P.

Example 3.1. Let P be the polytope of real doubly stochastic $n \times n$ matrices. If $v_1, v_2, \dots, v_n$ are the $n \times n$ permutation matrices corresponding to the elements of the cyclic subgroup of the symmetric group generated by the cycle $(1\ 2\ \cdots\ n)$ (or any n permutation matrices with pairwise disjoint supports) then $v_1, v_2, \dots, v_n$ are the vertices of a special simplex in P. Indeed, each facet of P is defined by an equation of the form $x_{ij} = 0$ in $\mathbb{R}^{n \times n}$ and misses exactly one of $v_1, v_2, \dots, v_n$.

Example 3.2. Let Ω be a poset on the ground set $[m] := \{1, 2, \dots, m\}$ which is graded of rank $n-2$ (we refer to [24, Chapter 3] for basic background and terminology on partially ordered sets) and $P = O(\Omega)$ be the order polytope of Ω in $\mathbb{R}^{m+1}$. Let Ω^0 be the poset obtained from Ω by adjoining a minimum element 0 and for $1 \leq i \leq n$ let v_i be the characteristic vector of the ideal of elements of Ω^0 of rank at most $i-1$, so that v_i is a vertex of P. Since a facet of P is defined either by an equation of the form $x_i = x_j$ with $i < j$ in Ω^0 and i, j in successive ranks or by one of the form $x_i = 0$ for $i \in \Omega^0$ of rank $n-1$, it follows that $v_1, v_2, \dots, v_n$ are the vertices of a special simplex in P.

Lemma 3.3. *Suppose that $v_1, v_2, \dots, v_n$ are the vertices of a special simplex in P. If F is a face of P of codimension k for some $1 \leq k \leq n-1$ and F does not contain any of $v_1, v_2, \dots, v_k$ then F must contain v_i for all $k+1 \leq i \leq n$.*

Proof. Let Σ be the special simplex with vertices $v_1, v_2, \dots, v_n$. Any codimension k face of a polytope can be written as the intersection of k facets, so we can write $F = F_1 \cap F_2 \cap \cdots \cap F_k$ where the F_j are facets of P. For each $1 \leq i \leq k$ we have $v_i \notin F$ and hence $v_i \notin F_j$ for some $j = j_i$. Since Σ is special the integers $j_1, j_2, \dots, j_k$ are all distinct and hence for each $1 \leq j \leq k$ we have $v_i \notin F_j$ for some $1 \leq i \leq k$, which in turn implies that $v_i \in F_j$ for all $k+1 \leq i \leq n$. It follows that $v_i \in F_1 \cap F_2 \cap \cdots \cap F_k = F$ for all $k+1 \leq i \leq n$. □

Lemma 3.4. *Suppose that $\tau = (v_p, v_{p-1}, \dots, v_1)$ is an ordering of the vertices of P such that $\sigma = \{v_1, v_2, \dots, v_n\}$ is the vertex set of a special simplex in P. Let Δ be*

the abstract simplicial complex on $\{v_{n+1}, \ldots, v_p\}$ *defined by the reverse lexicographic triangulation of* $\mathcal{F}(P) \backslash \sigma$ *with respect to* $(v_p, v_{p-1}, \ldots, v_{n+1})$.

(i) *The reverse lexicographic triangulation* Δ_τ *of* P *is abstractly isomorphic to the simplicial join* $\sigma * \Delta$.

(ii) Δ *is abstractly isomorphic to the boundary complex of a simplicial polytope of dimension* $m - n + 1$.

Proof. (i) Let $\sigma_i = \{v_1, \ldots, v_i\}$ for $0 \leq i \leq n$, so that $\sigma_0 = \emptyset$ and $\sigma_n = \sigma$, and let Δ_i denote the abstract simplicial complex on the set $\{v_{i+1}, \ldots, v_p\}$ defined by the reverse lexicographic triangulation of $\mathcal{F}(P) \backslash \sigma_i$ with respect to the ordering $(v_p, v_{p-1}, \ldots, v_{i+1})$. To prove that $\Delta_0 = \sigma_n * \Delta_n$, which is the assertion in the lemma, we will prove that $\mathcal{F}(P) \backslash \sigma_i$ is pure $(m - i)$-dimensional and that $\Delta_0 = \sigma_i * \Delta_i$ for all $0 \leq i \leq n$ by induction on i. This is obvious for $i = 0$ so let $1 \leq i \leq n$. By induction, any maximal face F of $\mathcal{F}(P) \backslash \sigma_{i-1}$ is a codimension $i - 1$ face of P. Since F does not contain any of the vertices $v_1, \ldots, v_{i-1}$, by Lemma 3.3 we have $v_i \in F$. This implies that $\mathcal{F}(P) \backslash \sigma_i$ is pure $(m - i)$-dimensional and that $\Delta_{i-1} = v_i * \Delta_i$. The last equality and the induction hypothesis $\Delta_0 = \sigma_{i-1} * \Delta_{i-1}$ imply that $\Delta_0 = \sigma_i * \Delta_i$, which completes the induction.

(ii) Let V be the linear subspace of $\mathbb{R}^q$ parallel to the affine span of the vertices in σ and P/V be the corresponding quotient polytope of P, so that P/V has dimension $m - n + 1$. Part (i) and Proposition 2.3 imply that Δ is abstractly isomorphic to a reverse lexicographic triangulation of the boundary complex of P/V. This is in turn isomorphic to the boundary complex of a simplicial polytope of dimension $m - n + 1$ by Lemma 2.1. $\square$

The following theorem is the key to the results in this paper.

Theorem 3.5. *Suppose that* P *is an integer polytope and* $\tau = (v_p, v_{p-1}, \ldots, v_1)$ *is an ordering of its vertices such that:*

(i) τ *is compressed and*

(ii) $\{v_1, v_2, \ldots, v_n\}$ *is the vertex set of a special simplex in* P.

Then

$$\sum_{r \geq 0} \text{Ehrhart}(P, r)\, t^r = \frac{h(t)}{(1 - t)^{m+1}}$$

where $h(t) = h_0 + h_1 t + \cdots + h_d t^d$ *is the* h*-polynomial of the boundary complex of a simplicial polytope* Q *of dimension* $d = m - n + 1$*, so that* $h(t)$ *satisfies the conditions in the* g*-theorem.*

In particular $h_i = h_{d-i}$ *for all* i *and* $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$.

Moreover, Q *can be chosen so that its boundary complex is abstractly isomorphic to the reverse lexicographic triangulation of* $\mathcal{F}(P) \backslash \{v_1, \ldots, v_n\}$ *with respect to the ordering* $(v_p, v_{p-1}, \ldots, v_{n+1})$.

Proof. Let $\sigma = \{v_1, v_2, \ldots, v_n\}$ and let Δ denote the reverse lexicographic triangulation of $\mathcal{F}(P) \backslash \sigma$ with respect to the ordering $(v_p, v_{p-1}, \ldots, v_{n+1})$. Lemma 2.2 guarantees that the proposed equation holds with $h(t) = h(\Delta_\tau, t)$. Part (i) of Lemma 3.4 implies that

$$h(\Delta_\tau, t) = h(\sigma * \Delta, t) = h(\sigma, t)\, h(\Delta, t) = h(\Delta, t),$$

since face complexes of simplices have h-polynomial equal to 1, and the result follows from part (ii) of the same lemma. $\square$

We now apply Theorem 3.5 to the Birkhoff polytope and to order polytopes of graded posets. Observe that our theorem does not apply to all integer polytopes since 0-1 polytopes with no regular unimodular triangulations are known to exist [14].

Magic squares and the Birkhoff polytope. Let P be the polytope of real doubly stochastic $n \times n$ matrices. Observe that the polynomial $\mathrm{Ehrhart}(P, r)$ coincides with the function $H_n(r)$ of Theorem 1.1. Since P is a compressed integer polytope of dimension $(n-1)^2$, Theorem 3.5 and Example 3.1 imply immediately the following corollary.

Corollary 3.6. *For any positive integer n we have*

$$\sum_{r \geq 0} H_n(r)\, t^r = \frac{h(t)}{(1-t)^{(n-1)^2+1}}$$

where $h(t) = h_0 + h_1 t + \cdots + h_d t^d$ is the h-polynomial of the boundary complex of a simplicial polytope of dimension $d = n^2 - 3n + 2$, so that $h(t)$ satisfies the conditions in the g-theorem.

In particular $h_i = h_{d-i}$ for all i and $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$.

In view of the last statement in Corollary 4.1, the polytope in the previous corollary can be constructed by pulling in an arbitrary order the vertices of the quotient of P with respect to the affine span of the vertices $v_1, v_2, \ldots, v_n$, chosen explicitly as in Example 3.1.

Eulerian polynomials and equatorial spheres. Let Ω be a graded poset on the ground set $[m] := \{1, 2, \ldots, m\}$ of rank $n-2$. Let Ω_i be the set of elements of Ω of rank $i-1$ for $1 \leq i \leq n-1$ and $\mathcal{L}(\Omega)$ be the set of *linear extensions* of Ω, meaning the set of permutations $w = (w_1, w_2, \ldots, w_m)$ of $[m]$ for which $w_i <_\Omega w_j$ implies $i < j$. We assume that Ω is *naturally labeled*, meaning that the identity permutation $(1, 2, \ldots, m)$ is a linear extension. The Ω-*Eulerian polynomial* is defined as

$$W(\Omega, t) = \sum_{w \in \mathcal{L}(\Omega)} t^{\mathrm{des}(w)}$$

where

$$\mathrm{des}(w) = \#\{i \in [m-1] : w_i > w_{i+1}\}$$

is the number of descents of w. Following [15] we call a function $g : \Omega \to \mathbb{R}$ *equatorial* if $\min_{a \in \Omega} g(a) = 0$ and for each $2 \leq i \leq n-1$ there exist $a_{i-1} \in \Omega_{i-1}$ and $a_i \in \Omega_i$ such that $a_{i-1} <_\Omega a_i$ and $g(a_{i-1}) = g(a_i)$. An ideal I or, more generally, a strictly increasing chain of ideals $I_1 \subset I_2 \subset \cdots \subset I_k$ in Ω is *equatorial* if the characteristic function χ_I of I or the sum $\chi_{I_1} + \chi_{I_2} + \cdots + \chi_{I_k}$, respectively, is equatorial. The *equatorial complex* $\Delta_{eq}(\Omega)$, introduced in [15], is the abstract simplicial complex on the vertex set of equatorial ideals of Ω whose simplices are the equatorial chains of ideals in Ω.

The following theorem is proved in Corollary 3.8 and Theorem 3.14 of [15].

Theorem 3.7. (Reiner–Welker [15]) *Let Ω be a naturally labeled, graded poset on $[m]$ having $n-1$ ranks. The equatorial complex $\Delta_{eq}(\Omega)$ is abstractly isomorphic to the boundary complex of a simplicial polytope of dimension $d = m - n + 1$ which has h-polynomial equal to the Ω-Eulerian polynomial $W(\Omega, t)$.*

Hence $W(\Omega, t)$ satisfies the conditions in the g-theorem and, in particular, it has symmetric and unimodal coefficients.

Let P be the order polytope of Ω and Ω^0 be the poset obtained from Ω by adjoining a minimum element 0. Recall that the vertices of P are the characteristic vectors of the nonempty ideals of Ω^0. The order polytope comes with its canonical triangulation [25] [15, Proposition 2.1], which is a unimodular triangulation with maximal simplices bijecting to the linear extensions of Ω. This canonical triangulation is in fact the reverse lexicographic triangulation of $O(\Omega)$ with respect to any ordering $(u_p, u_{p-1}, \dots, u_1)$ of its vertices such that $i < j$ whenever the ideal of Ω^0 defined by u_i is strictly contained in that defined by u_j. We will use the following lemma.

Lemma 3.8. *Let v_i be the characteristic vector of the ideal of elements of Ω^0 of rank at most $i-1$ for $1 \le i \le n$. Let $\sigma = \{v_1, \dots, v_n\}$ and $\tau = (v_p, \dots, v_{n+1})$ be an ordering of the remaining vertices of P such that $i < j$ whenever $i, j \ge n+1$ and the ideal defined by v_i is strictly contained in that defined by v_j.*

The equatorial complex $\Delta_{eq}(\Omega)$ is the abstract simplicial complex defined by the reverse lexicographic triangulation of $\mathcal{F}(P) \setminus \sigma$ with respect to τ.

Proof. Let $\mathcal{F}$ denote the face complex of P and let $x_{\hat{1}} = 0$ by convention. The maximal faces of $\mathcal{F} \setminus \sigma$ are the faces of P defined by systems of equations of the form $x_{i_s} = x_{j_s}$ for $0 \le s \le n-1$ where (i) $i_0 = 0$ and $j_{n-1} = \hat{1}$, (ii) $i_s \in \Omega_s$ for $1 \le s \le n-1$, $j_s \in \Omega_{s+1}$ for $0 \le s \le n-2$ and $i_s <_\Omega j_s$ for $1 \le s \le n-2$ and (iii) if $j_s = i_{s+1}$ for consecutive values $s = a, a+1, \dots, b-1$ of s then the interval $[i_a, j_b]$ in $\hat{\Omega}$ consists only of the elements of the chain $i_a < i_{a+1} < \cdots < i_b < j_b$. The statement of the Lemma follows from the description of the maximal faces of a reverse lexicographic triangulation $\Delta(\mathcal{F})$ (see Section 2) and that of the maximal faces of $\Delta_{eq}(\Omega)$ (see [15, Proposition 3.5]). We omit the details. $\square$

Proof of Theorem 3.7. Let P be the order polytope of Ω, as before. Observe that $\mathrm{Ehrhart}(P, r)$ is equal to the number of order reversing maps $\rho : \Omega \to \{0, 1, \dots, r\}$. It follows from [24, Theorem 4.5.14] that

$$\sum_{r \ge 0} \mathrm{Ehrhart}(P, r)\, t^r = \frac{W(\Omega, t)}{(1-t)^{m+1}}. \tag{4}$$

Let the vertices $v_1, v_2, \dots, v_p$ of P and $\tau = (v_p, \dots, v_{n+1})$ be as in Lemma 3.8. We checked in Example 3.2 that $v_1, v_2, \dots, v_n$ are the vertices of a special simplex in P. Since P is a compressed integer polytope (see Section 2) Theorem 3.5 applies and we have

$$\sum_{r \ge 0} \mathrm{Ehrhart}(P, r)\, t^r = \frac{h(t)}{(1-t)^{m+1}},$$

where $h(t) = h_0 + h_1 t + \cdots + h_d t^d$ is the h-polynomial of a simplicial polytope of dimension $d = m - n + 1$ having, in view of Lemma 3.8, boundary complex abstractly isomorphic to $\Delta_{eq}(\Omega)$. Comparison with (4) yields $h(t) = W(\Omega, t)$ and completes the proof. $\square$

4. Rational polyhedral cones

In this section we state several corollaries of Theorem 3.5, including a generalization of Corollary 3.6 to magic labelings of bipartite graphs. Let Φ be a $p \times q$ integer matrix with rank p and E_Φ be the monoid of vectors $x \in \mathbb{N}^q$ satisfying the homogeneous system of linear equations $\Phi\, x = 0$. We assume that E_Φ is nonzero, so that $q \geq p+1$, and set $m = q - p - 1$. We denote by $\mathcal{C}_\Phi$ the cone of vectors $x = (x_1, x_2, \dots, x_q) \in \mathbb{R}^q$ satisfying $\Phi\, x = 0$ and $x_i \geq 0$ for all i. Let $L(x) = a_1 x_1 + a_2 x_2 + \cdots + a_q x_q$ be a linear functional on $\mathbb{R}^q$ with $a_i \in \mathbb{Q}$ for all i such that the set

$$P = \{x \in \mathcal{C}_\Phi : L(x) = 1\} \tag{5}$$

is nonempty and bounded. Thus P is an m-dimensional convex polytope. Observe that

$$\mathrm{Ehrhart}(P, r) = \#\,\{x \in E_\Phi : L(x) = r\} \tag{6}$$

for any nonnegative integer r. Let $\bar{E}_\Phi$ be the submonoid of E_Φ consisting of those elements having positive coordinates. We say that $\beta \in \bar{E}_\Phi$ is the *unique minimal element* of $\bar{E}_\Phi$ if we have $\beta \leq \gamma$ coordinatewise for all $\gamma \in \bar{E}_\Phi$.

Corollary 4.1. *Let P be an m-dimensional polytope, as in* (5). *If*

- (i) *P is an integer polytope,*
- (ii) *$\bar{E}_\Phi$ has a unique minimal element β and*
- (iii) *there exists a compressed ordering $\tau = (v_p, v_{p-1}, \dots, v_1)$ of the vertices of P such that $v_1 + v_2 + \cdots + v_n = \beta$ for some n*

then the conclusion of Theorem 3.5 holds.

Proof. In view of Theorem 3.5 it suffices to show that $\{v_1, v_2, \dots, v_n\}$ is the vertex set of a special simplex in P. Let $\beta = (\beta_1, \beta_2, \dots, \beta_q)$ and let F be a facet of P, so that F is defined by an equation of the form $x_k = 0$ for some $1 \leq k \leq q$. We need to show that exactly one of $v_1, v_2, \dots, v_n$ has positive kth coordinate. Clearly at least one of $v_1, v_2, \dots, v_n$ has this property, since $\beta_i > 0$ for all i. Assume on the contrary that at least two v_j have positive kth coordinate, say v_1 and v_2, so that $1 \leq \gamma_k < \beta_k$ if $v_1 = (\gamma_1, \gamma_2, \dots, \gamma_q)$. Since F is a facet of P there exists a point $x = (x_1, x_2, \dots, x_q)$ in the affine span of P, which we may assume to be rational, satisfying $x_k < 0$ and $x_i > 0$ for all $i \neq k$. By replacing x with a suitable integer multiple we find a point $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_q) \in \mathbb{Z}^q$ satisfying $\Phi\, \alpha = 0$, $\alpha_k < 0$ and $\alpha_i > 0$ for all $i \neq k$. We may choose a nonnegative integer t so that $0 < \alpha_k + \beta_k + t\gamma_k < \beta_k$ (with $t = 0$ if $\alpha_k + \beta_k > 0$). Then $\alpha + \beta + t v_1$ is in $\bar{E}_\Phi$ and has kth coordinate strictly less than β_k, which contradicts the minimality of β. □

The condition that $\bar{E}_\Phi$ has a unique minimal element is satisfied if $(1, 1, \dots, 1) \in \mathbb{R}^q$ is in E_Φ and is known to hold if and only if the semigroup ring $R_\Phi = \mathbf{k}[E_\Phi]$, generated over a field $\mathbf{k}$ by the monomials corresponding to elements of E_Φ, is Gorenstein; see for instance [23, Section I.13].

Corollary 4.2. *Let P be an m-dimensional polytope, as in* (5). *If*

- (i) *P is a compressed integer polytope,*
- (ii) *R_Φ is Gorenstein and*
- (iii) *E_Φ is generated as a monoid by the vertices of P*

then the conclusion of Theorem 3.5 holds where, in the statement of the theorem, n has the value $L(\beta)$ for the unique minimal element β of $\bar{E}_\Phi$.

Proof. Let β be the unique minimal element of $\bar{E}_\Phi$, whose existence is guaranteed by (ii). Then (iii) implies that $\beta = v_1 + v_2 + \cdots + v_n$ for some vertices $v_1, v_2, \ldots, v_n$ of P, which must be pairwise distinct. Because of (i) any ordering $(v_p, v_{p-1}, \ldots, v_1)$ of the vertices of P satisfies the assumptions of Corollary 4.1. The result follows from this corollary observing that $L(\beta) = n$. $\square$

General conditions on Φ and P which guarantee assumptions (i) and (iii) of Corollary 4.2 were given by Ohsugi and Hibi [13].

Corollary 4.3. *Let P be an m-dimensional polytope, as in* (5)*. If P is a* 0-1 *polytope and R_Φ is Gorenstein then the conclusion of Corollary 4.2 holds.*

Proof. Since P is a 0-1 polytope, Theorem 1.1 and Lemma 2.1 in [13] imply, respectively, that P is compressed and that E_Φ is generated as a monoid by the integer points of P, which are exactly the vertices of P. Thus the assumptions of Corollary 4.2 hold. $\square$

Remark 4.4. Recall that a matrix is called *totally unimodular* if all its subdeterminants are equal to $0, -1$ or 1. Suppose that the linear functional $L(x) = a_1x_1 + a_2x_2 + \cdots + a_qx_q$ satisfies $a_i \in \mathbb{Z}$ for all i. It follows from the results of [7] that P is an integer polytope if the matrix obtained from Φ by adding the row $(a_1, a_2, \ldots, a_q)$ is totally unimodular. Hence, in view of Corollary 4.3, this statement and the assumptions that (i) $P \subseteq [0,1]^q$ and (ii) R_Φ is Gorenstein imply the conclusion of Corollary 4.1.

Magic labelings of graphs. Let G be a graph (multiple edges and loops allowed) with p vertices and q edges and edge set $\mathcal{E}$. A *magic labeling* [17] of G of *index* r is an assignment $\ell : \mathcal{E} \to \mathbb{N}$ of nonnegative integers to the edges of G such that for each vertex v of G the sum of the labels of all edges incident to v is equal to r, in other words,

$$\sum_{e:\ v \in e} \ell(e) = r.$$

Let $\mathbb{R}^{\mathcal{E}}$ denote the real vector space with basis $\mathcal{E}$ and let x_e be the linear functional on $\mathbb{R}^{\mathcal{E}}$ dual to the basis element $e \in \mathcal{E}$. Let Φ be a full rank, $p' \times q$ integer matrix with kernel the set of $x \in \mathbb{R}^{\mathcal{E}}$ satisfying the linear system of equations of the form

$$\sum_{e:\ v \in e} x_e = \sum_{e':\ v' \in e'} x_{e'} \tag{7}$$

in $\mathbb{R}^{\mathcal{E}}$, where v, v' are vertices of G. If $L(x)$ is any of the functionals on $\mathbb{R}^{\mathcal{E}}$ in (7) and P is as in (5) then $\mathrm{Ehrhart}(P, r)$ counts the number $H_G(r)$ of magic labelings of G of index r. It follows from [17, Proposition 2.9] and either [22, Theorem 2.3] (applied as in [22, Example 2.4 (b)] in the case of the Birkhoff polytope) or [13, Theorem 1.1] that conditions (i) and (iii) of Corollary 4.2 are both satisfied if the graph G is bipartite (or, more generally, if it satisfies condition (iii) of [17, Proposition 2.9]). Clearly condition (ii) is satisfied if G is regular, since then $(1, 1, \ldots, 1) \in E_\Phi$. Assuming further that G is connected we have $p' = p - 2$. The following corollary specializes to Corollary 3.6 when G is the complete bipartite graph on two sets of vertices, each of size n.

Corollary 4.5. *For $n \geq 1$ and for any connected regular bipartite graph G with p vertices and $q = np/2$ edges we have*

$$\sum_{r \geq 0} H_G(r)\, t^r = \frac{h(t)}{(1-t)^{m+1}}$$

where $m = q - p + 1$ and $h(t) = h_0 + h_1 t + \cdots + h_d t^d$ is the h-polynomial of the boundary complex of a simplicial polytope of dimension $d = m - n + 1$, so that $h(t)$ satisfies the conditions in the g-theorem.

In particular $h_i = h_{d-i}$ for all i and $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$.

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